Math 206A Lecture 9 Notes

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1 Borsuk's Conjecture: The Final Chapter

1.1 Remaining lemmas

Let's finish the proof of the Kahn-Kalai theorem. Our main lemma is the following.

Lemma 1.1. *Let $M = \{(x_1, \dots, x_n) \subseteq \{\pm 1\}^n : x_1 = 1, x_2 \cdots x_n = 1\}$, and let $A \subseteq M$ be such that $a \cdot a' \neq 0$ for all $a, a' \in A$. Then $|A| < c^n$ for some $c < 2$.*

Proof. Last time we had $G(t) = (t-1)(t-2)\cdots(t-p+1)$ with $t \in \mathbb{N}$. For $\overline{a} \in A$, $\overline{z} = (1, z_2, \dots, z_n)$, think of $G(\overline{a} \cdot \overline{z})$ as a polynomial in the z_i of degree $p-1 < n/4$. Let F_a be the square-free part of $G(\overline{a} \cdot \overline{z})$. For example, if $\overline{a} = (1, 1, -1, -1, 1)$, $n = 5$ and $p = 3$, then

$$\begin{aligned} F(\overline{z} \cdot \overline{z}) &= (1 + z_2 - z_3 - z_4 + z_5 - 1)(1 + z_2 - z_3 - z_4 + z_5 - 2) \\ &= 1 + z_2^2 + x_3^2 - x_2 z_3 + z_4^2 - z_2 z_4 + z_3 z_4 + \cdots . \end{aligned}$$

Then $F_a = 1 - z_2 z_4 - z_2 z_4 + z_3 z_4 + \cdots$. $\square$

We need a lemma for our lemma.

Lemma 1.2 (independence lemma)**.** *The set $\{F_a : a \in A\}$ are linearly independent.*

Proof. Note that $t \neq 0 \pmod p \iff G(t) = 0 \mod p$. Proceed by contradiction, assuming $\lambda_1 F_{\overline{a}_1} + \lambda_2 F_{\overline{a}_2} + \cdots = 0$ with $\lambda_1 \neq 0 \pmod p$. Then $G(\overline{a}_1 \cdot \overline{a}_1) = G(n) = G(4p) \neq 0 \pmod p$. So $F_{\overline{a}_1} \neq 0 \pmod p$. Also note that $G(a \cdot a') = F_a(a')$ for all $a, a' \in M$. We also have that $F_{\overline{a}}(\overline{a}') = 0 \pmod p$ for $a' \neq a$. Together, these two imply the independence lemma. Indeed, substitute $z = \overline{a}_1$ into the linear combination to get $\lambda_1 F_{\overline{a}_1}(\overline{a}_1) + 0 + \cdots + 0 = 0 \pmod p$. Since $F_{\overline{a}_1}(\overline{a}_1) \neq 0 \pmod p$, $\lambda_1 = 0 \pmod p$. We claim that $a \cdot a' = 0 \pmod 4$. Do this as an exercise. This means that $a \cdot a' = 0 \pmod 4 \implies F_{\overline{a}_1}(\overline{a}_1) = 0 \pmod p$. Combining these results proves the lemma. $\square$

We return to the main lemma.

Proof. So $|A| < \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n/4} < (n/4+1)\binom{n}{n/4} < c^n$. This implies the main lemma. $\square$

So we can finish the proof of the theorem:

Proof. The lemma implies that M cannot be partitioned into fewer than $2^{n-2}/c^n \gg (n^2+1)$ parts with no $a \cdot a' = 0$. This implies that, for large enough n, $M \otimes M$ cannot be partitioned into fewer than (n^2+1) parts of smaller diameter. The n^2+1 comes from the fact that $\dim(M \otimes M) = n^2$. $\square$

1.2 Aftermath

Professor Pak believes that Borsuk's conjucture probably fails for $n = 4$ or $n = 5$. There is no reason why we need the large construction in the Kahn-Kalai proof. It is known (from 2016) that Borsuk's conjecture fails in dimension 64.

One can ask about the chromatic number $\chi_1(\mathbb{R}^d)$ of the unit distance graph. It is known that $5 \leq \chi_1(\mathbb{R}^2) \leq 7$. How does this behave asymptotically?

Theorem 1.1 (Franklin-Wilson). $c^d \leq \chi_1(\mathbb{R}^d) \leq d^d$ *for some constant c.*